NEW FAMILIES OF ODD HARMONIOUS GRAPHS

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ABSTRACT

In this paper, we show that the number of edges for any odd harmonious Eulerian graph is congruent to 0 or 2 (mod 4), and we found a counter example for the inverse of this statement is not true. We also proved that, the graphs which are constructed by two copies of even cycle Cn sharing a common edge are odd harmonious. In addition, we obtained an odd harmonious labeling for the graphs which are constructed by two copies of cycle Cn sharing a common vertex when n is congruent to 0 (mod 4). Moreover, we show that, the Cartesian product of cycle graph Cm and path Pn for each $n \geq 2,\ m \equiv 0 \pmod 4$ *are odd harmonious graphs. Finally many new families of odd harmonious graphs are introduced.*

KEYWORDS

Odd harmonious labeling, Eulernian graph, Cartesian product, Cyclic graphs.

INTRODUCTION

Graph labeling have often been motivated by practical problems is one of fascinating areas of research. A systematic study of various applications of graph labeling is carried out in Bloom and Golomb [1]. Labeled graph plays vital role to determine optimal circuit layouts for computers and for the representation of compressed data structure. Many of the results about graph labeling are collected and updated regularly in a survey by Gallian [2]. The reader can consult this survey for more information about the subject.

We begin simple, finite, connected and undirected graph G = (V(G), E(G)) with p vertices and q edges. For all other standard terminology and notations we follow Harary [3].

Most graph labeling methods trace their origion to one introduced by Rosa [4] called such a labeling a β-valuation and Golomb [5] subsequently called graceful labeling, and one introduced by Graham and Sloane [6] called harmonious labeling. Several infinite families of graceful and harmonious graphs have been reported. Many illustrious works on graceful graphs brought a tide to different ways of labeling the elements of graph such as odd graceful.

A graph G of size q is odd-graceful, if there is an injection f from $V(G)$ to $\{0,1,2,\ldots, 2q-1\}$ such that, when each edge xy is assigned the label or weight $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, 2q-1\}$. This definition was introduced by Gnanajothi [7]. Many researchers have studied odd graceful labeling. Seoud and Abdel-Aal [8], [9] they give a survey of all connected graph of order ≤ 6 which are odd graceful, and they also introduce some families of odd graceful graphs.

A graph G is said to be odd harmonious if there exists an injection $f: V(G) \rightarrow \{0, 1, 2, \ldots, 2q-1\}$ such that the induced function $f^*: E(G) \rightarrow \{1, 3, \ldots, 2q-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. Then f is said to be an odd harmonious labeling of G.

Liang and Bai [10] introduced concept of odd harmonious labeling and they have obtained the necessary conditions for the existence of odd harmonious labeling of a graph. They proved that if G is an odd harmonious graph, then G is a bipartite graph. Also they claim that if a $(p, q)-$ graph G is odd harmonious, then $2\sqrt{q} \leq p \leq 2q-1$, but this is not always correct. Take the path P_2 as a counter example.

In this paper, we show that the number of edges for any odd harmonious Eulernian graph must be congruent to 0 or 2 (mod 4), and we found a counter example to prove that, not necessary every Eulernian graph, with number of edges congruent to 0 or 2 (mod 4), to be an odd harmonious graph. This result corresponds to the result in case G is graceful and Eulerian, which had been stated and proved by rosa [4]. Also we show that many new families of graphs are odd harmonious. For instance, we obtained the odd harmonious labelings for joining two copies of even cycles with a common edge or with a common vertex and $C_n \times P_m$. Brief, new families of odd harmonious graphs are introduced.

MAIN RESULTS

Theorem 2.1

If G is an odd harmonious Eulerian graph with q edges, then $q \equiv 0$ or 2 *(mod 4).*

Proof

Let G be an odd harmonious Eulerian graph, and let $f: V(G) \rightarrow \{0, 1, 2, \ldots, 2q-1\}$ be an odd harmonious labeling for G. Since G is an Eulerian graph then $\sum (f(v_i) + f(v_j)) = 2k$, k is a constant. For each v_i, $v_j \in V(G)$, $\sum f(v_i) + f(v_j) = 2k'$, so $1+3+5+\ldots+2q-1 = 2k'$, this implies that $\frac{q}{2}(1+2q-1) = 2k'$. Hence $q \equiv 0$ *or 2* (*mod 4*).

The inverse of the last result is not true.

Remark 2.2

Not necessary every Eulernian graph with number of edges congruent to 0 or 2 (mod 4) to be an odd harmonious graph.

Proof

By counter example: let $G = C_6$ where C_6 is Eulerian graph with $q = 6$, $q \equiv 2$ *(mod 4)*, while C_6 is not odd harmonious follows from, every C_n is odd harmonious if and only if $n \equiv 0$ (*mod 4*), [10].

Theorem 2.3

Two copies of even cycle C_n sharing a common edge is an odd harmonious graph.

Proof

Let $v_1, v_2, ..., v_n$ be the vertices of cycle C_n of even order. Consider two copies of cycle C_n. Let G denotes the graph of two copies with even cycle C_n sharing a common edge, clearly G has $|V(G)| = 2n-2$ and $|E(G)| = 2n-1$. Without loss of generality let this edge be $e = v_{\frac{n}{2}+1}\ v_{\frac{3n}{2}}$.

We define the labeling function:

$$f : V(G) \to \{0, 1, 2, 3, \ldots, 4n-3\}$$

as follows, we consider two cases:

Case (i): $n \equiv 0 \ (mod\ 4)$,

for $1 \le i \le \frac{n}{2}+1$:

$$f(v_i) = (i-1),$$

for $\frac{n}{2}+2 \le i \le n-1$ (we ignore this step when $n = 4$):

$$f(v_i) = \begin{cases} i+1, & (i \text{ even}), \quad i = \frac{n}{2}+2, \frac{n}{2}+4, ..., n-2 \\ i-1, & (i \text{ odd}), \quad i = \frac{n}{2}+3, \frac{n}{2}+5, ..., n-1, \end{cases}$$

for $n \le i \le \frac{3n}{2}$:

$$f(v_i) = 3n - i - 1, ,$$

and

for $\frac{3n}{2}+1 \le i \le 2n-2$ (we ignore this step when $n = 4$):

$$f(v_i) = \begin{cases} 3n-i-3, & (i \text{ odd }), \quad i = \frac{3n}{2}+1, \frac{3n}{2}+3, ..., 2n-1 \\ 3n-i-1, & (i \text{ even }), \quad i = \frac{3n}{2}+2, \frac{3n}{2}+4, ..., 2n-2. \end{cases}$$

Case (ii): $n \equiv 2 \ (mod\ 4)$,

for $1 \le i \le \frac{n}{2}+1$:

$$f(v_i) = (i-1),$$

for $\frac{n}{2}+2 \le i \le n+1$:

$$f(v_i)=\begin{cases} i+1, & (i \text{ odd}), \quad i=\frac{n}{2}+2,\frac{n}{2}+4,\dots,n+1 \\ i-1, & (i \text{ even}), \quad i=\frac{n}{2}+3,\frac{n}{2}+5,\dots,n, \end{cases}$$

for $n+2 \le i \le 2n-2$:

$$f(v_i)=\begin{cases} 3n-i-1, & (i \text{ even}), \quad i=n+2,n+4,\dots,2n-2, \\ 3n-i+3, & (i \text{ even}), \quad i=n+3,n+5,\dots,\frac{3n}{2}, \\ 3n-i+1, & (i \text{ odd}), \quad i=\frac{3n}{2}+2,\frac{3n}{2}+4,\dots,2n-1. \end{cases}$$

We observe that f is injective.

The edge labels will be as follows:

Case (i): $n \equiv 0 \ (mod\ 4)$,

- The vertices v_i and v_{i+1}, $1 \le i \le \frac{n}{2}$ induce the edge labels:

$f^*(v_i\, v_{i+1}) = f(v_i) + f(v_{i+1}) = \{2i-1,\ 1 \le i \le \frac{n}{2}\} = \{1,3,5,\ \dots,\ n-1\}$.

- The vertices v_1 and v_{2n-2}, induce the edge labels:

$f^*(v_1\, v_{2n-2}) = f(v_1) + f(v_{2n-2}) = n+1$.

- The vertices $v_{\frac{n}{2}+1}$ and $v_{\frac{n}{2}+2}$, induce the edge labels:

$f^*(v_{\frac{n}{2}+1}\, v_{\frac{n}{2}+2}) = f(v_{\frac{n}{2}+1}) + f(v_{\frac{n}{2}+2}) = n+3$.

- The vertices v_i and v_{i+1}, $\frac{n}{2}+2 \le i \le n-2$ induce the edge labels:

$f^*(v_i\, v_{i+1}) = f(v_i) + f(v_{i+1}) = \{2i+1,\ \frac{n}{2}+2 \le i \le n-2\}$

$= \{n+5,\ n+7, \dots,\ 2n-3\}$.

- The vertices $v_{\frac{n}{2}+1}$ and $v_{\frac{3n}{2}}$, induce the edge labels:

$f^*(v_{\frac{n}{2}+1}\, v_{\frac{3n}{2}}) = f(v_{\frac{n}{2}+1}) + f(v_{\frac{3n}{2}}) = n-1$.

- The vertices v_i and v_{i+1}, $\frac{3n}{2}+1 \le i \le 2n-3$ induce the edge labels:

$f^*(v_i\ v_{i+1}) = f(v_i) + f(v_{i+1}) = \{6n-2i-5,\ \frac{3n}{2}+1 \le i \le 2n-3\}$

$=\{3n\text{-}7,\ 3n\text{-}9,\ldots,\ 2n+1\}$.

- The vertices $v_{\frac{3n}{2}+1}$ and $v_{\frac{3n}{2}}$, induce the edge labels:

$f^*(v_{\frac{n}{2}+1}\ v_{\frac{3n}{2}}) = f(v_{\frac{n}{2}+1}) + f(v_{\frac{3n}{2}}) = 3n-5.$

- The vertices v_{n-1} and v_n, induce the edge labels:

$f^*(v_{n-1}\ v_n) = f(v_{n-1}) + f(v_n) = 3n-3.$

- The vertices v_i and v_{i+1}, $n \le i \le \frac{3n}{2}-1$ induce the edge labels:

$f^*(v_i\ v_{i+1}) = f(v_i) + f(v_{i+1}) = \{6n-2i-3,\ n \le i \le \frac{3n}{2}-1\}$

$=\{4n\text{-}3,\ 4n\text{-}5,\ldots,\ 3n-1\}$.

Case (ii): $n \equiv 2\ (mod\ 4)$,

- The vertices v_i and v_{i+1}, $1 \le i \le \frac{n}{2}$ induce the edge labels:

$f^*(v_i\ v_{i+1}) = f(v_i) + f(v_{i+1}) = \{2i-1,\ 1 \le i \le \frac{n}{2}\} = \{1,3,5,\ldots,\ n-1\}$.

- The vertices v_1 and v_{2n-2}, induce the edge labels:

$f^*(v_1\ v_{2n-2}) = f(v_1) + f(v_{2n-2}) = n+1.$

- The vertices $v_{\frac{n}{2}+1}$ and $v_{\frac{n}{2}+2}$, induce the edge labels:

$f^*(v_{\frac{n}{2}+1}\ v_{\frac{n}{2}+2}) = f(v_{\frac{n}{2}+1}) + f(v_{\frac{n}{2}+2}) = n+3.$

- The vertices v_i and v_{i+1}, $\frac{n}{2}+2 \le i \le n$ induce the edge labels:

$f^*(v_i\ v_{i+1}) = f(v_i) + f(v_{i+1}) = \{2i+1,\ \frac{n}{2}+2 \le i \le n\}$

$= \{n+5,\ n+7,\ldots,\ 2n+1\}$.

- The vertices $v_{\frac{n}{2}+1}$ and $v_{\frac{3n}{2}}$, induce the edge labels:

$f^*(v_{\frac{n}{2}+1}\ v_{\frac{3n}{2}}) = f(v_{\frac{n}{2}+1}) + f(v_{\frac{3n}{2}}) = 2n+3.$

- The vertices v_i and v_{i+1}, $\frac{3n}{2}+1 \le i \le 2n-3$ induce the edge labels:

$f^*(v_i\ v_{i+1}) = f(v_i) + f(v_{i+1}) = \{6n-2i-1,\ \frac{3n}{2}+1 \le i \le 2n-3\}$

$=\{3n\text{-}3,\ 3n\text{-}5,\ldots,\ 2n+5\}$.

- The vertices v_{n+1} and v_{n+2}, induce the edge labels:

$f^*(v_{n+1}\ v_n) = f(v_{n+1}) + f(v_n) = 3n\text{-}1.$

- The vertices v_i and v_{i+1}, $n+2 \le i \le \dfrac{3n}{2}$ induce the edge labels:

$$f^*(v_i\ v_{i+1}) = f(v_i) + f(v_{i+1}) = \{6n - 2i + 1,\ \ n+2 \le i \le \frac{3n}{2}\}$$

$$=\{4n\text{-}3,\ 4n\text{-}5,\ldots,\ 3n+1\}.$$

Now, we obtained all the edge labels {*1, 3, 5,…, 4n-3*} in each case, so f^* is injective as required. Hence *G* admits odd harmonious labeling.

Example 2.4. An odd harmonious labeling of two copies of cycles C_{12} sharing a common edge, and an odd harmonious labeling of two copies of cycles C_{10} sharing a common edge are shown in figure(1) and figure (2) respectively:

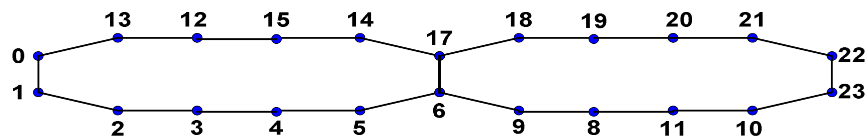


Figure (1): Two copies of cycle C_{12} sharing a common edge with its odd harmonious labeling.

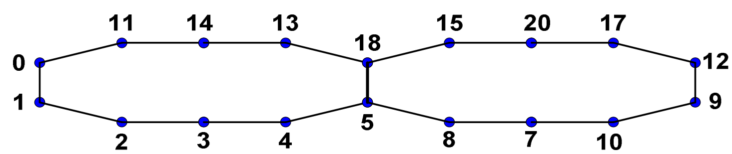


Figure (2): Two copies of cycle C_{10} sharing a common edge with its odd harmonious labeling.

In the following theorems we mention only the vertices labels, the reader can fulfill the proof as we did in the previous theorems.

Theorem 2.5

Two copies of even cycle C_n sharing a common vertex is an odd harmonious graph when $n \equiv 0 \pmod 4$.

Proof

Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n , $n \equiv 0 \ (mod\ 4)$. Consider *G* be the graph of two copies of C_n sharing a common vertex with $|V(G)| = 2n - 1$ and $|E(G)| = 2n$. Without loss of generality let this vertex be v_1. Let *G* be described as indicated in Figure (3).

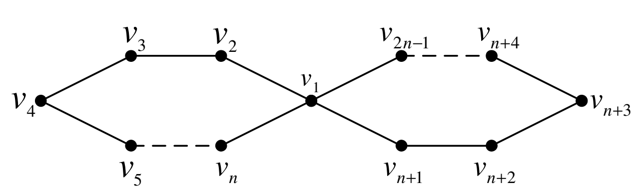


Figure (3)

We define the labeling function

$$f : V(G) \rightarrow \{0, 1, 2, 3, \ldots, 4n -1\}$$

as follows:

for $1 \leq i \leq \frac{n}{2}+1$:

$$f(v_i) = (i-1),$$

for $\frac{n}{2}+2 \leq i \leq n$:

$$f(v_i) = \begin{cases} i+1, & (i \text{ even }), \quad i = \frac{n}{2}+2, \frac{n}{2}+4, \ldots, n \\ i-1, & (i \text{ odd}), \quad i = \frac{n}{2}+3, \frac{3n}{2}+5, \ldots, n-1, \end{cases}$$

for $i = n+1$:

$$f(v_{n+1}) = 2n+3,$$

for $n+2 \leq i \leq 2n-1$:

$$f(v_i) = \begin{cases} i-2, & \text{when} \quad i = n+2,\ n+4, \ldots, \frac{3n}{2}, \\ i, & \text{when} \quad i = \frac{3n}{2}+2,\ \frac{3n}{2}+4, \ldots, 2n-1, \\ i+2, & \text{when} \quad i = n+3, n+5, \ldots, 2n-1. \end{cases}$$

Above defined labeling pattern exhausts all possibilities and in each case the graph under consideration admits odd harmonious labeling.

Example 2.6. An odd harmonious labeling of two copies of C_{12} sharing a common vertex is shown in figure(4).

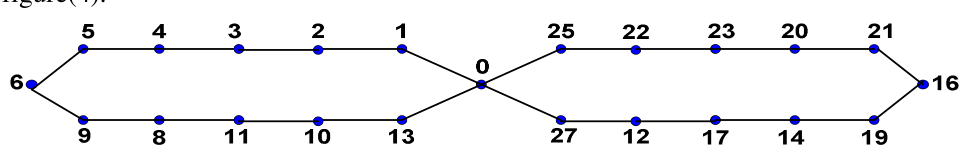


Figure (4): Two copies of cycle C_{12} sharing a common vertex with its odd harmonious labeling.

Remark 2.7

In theorems 2.3, 2.5 when n = 4, these theorems are coincided with corollary 3.12 (2) in [10] when i = 2,1 respectively.

In 1980 Graham and Sloane [6], proved that $C_m \times P_n$ is harmonious when n is odd. We generalized this result for odd harmonious labeling in the following theorem.

Theorem 2.8

The graphs $C_{4m} \times P_n$, for each $m \geq 1$, $n \geq 2$ are odd harmonious.

Proof

Let $G = C_{4m} \times P_n$ be described as indicated in Figure (5):

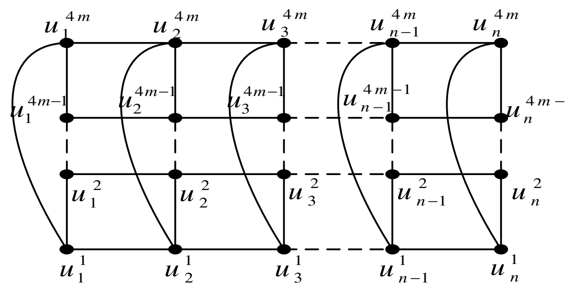


Figure (5)

The number of edges of the graph G is $4m(2n-1)$. We define the labeling function:

$f : V(G) \to \{0, 1, 2, 3, \ldots, 8m(2n-1)-1\}$

as follows:

$$f(u_i^1) = \begin{cases} 8m(i-1), & i = 1,3,5,\ldots,n \ \ or \ \ n-1, \\ 8mi - 4m + 1 & i = 2,4,6,\ldots,n-1 \ \ or \ \ n. \end{cases}$$

We consider the following three cases:

Case(i): $1 < j \leq \dfrac{4m}{2}$, $i = 1,3,5,\ldots,n \ or \ n-1$

$$f(u_i^j) = \begin{cases} 8mi - 4m - j + 3, & j = 2,4,6,\ldots,\dfrac{4m}{2}, \\ 8mi - 4m - j + 1, & j = 3,5,7,\ldots,\dfrac{4m}{2} - 1. \end{cases}$$

Case(ii): $1 < j \leq \dfrac{4m}{2}$, $i = 2,4,6,\ldots,n \ or \ n-1$

$$f(u_i^j) = \begin{cases} 8mi - 4m - j + 2, & j = 3,5,\ldots,\dfrac{4m}{2} - 1, \\ 8mi - 4m - j, & j = 2,4,6,\ldots,\dfrac{4m}{2}. \end{cases}$$

Case(iii): $\dfrac{4m}{2} + 1 \leq j \leq 4m$, $1 \leq i \leq n$

$$f(u_i^j) = \begin{cases} 8mi - 4m - j + 1, & i = 1,3,5,\ldots,n \ \ or \ \ n-1, \\ 8mi - 4m - j & i = 2,4,6,\ldots,n-1 \ \ or \ \ n. \end{cases}$$

Above defined labeling pattern exhausts all possibilities and in each case the graph under onsideration admits odd harmonious labeling.

Example 2.9. An odd harmonious labeling of the graph $C_4 \times P_5$ is shown in Figure (6).

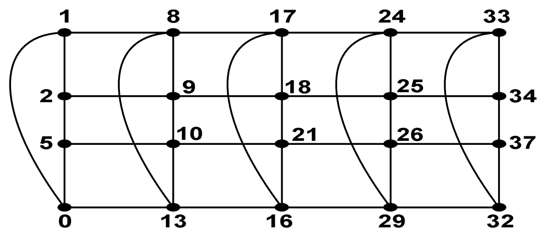


Figure (6): The graph $C_4 \times P_5$ with its odd harmonious labeling.

Let G_1 and G_2 be two disjoint graphs. The corona ($G_1 \Theta\, G_2$) of G_1 and G_2 is the graph obtained by taking one copy of G_1 (which has q_1 edges) and q_1 copies of G_2, and then sharing common edge between the i^{th} edge of G_1 and one edge in the i^{th} copy of G_2.

Theorem 2.10 The graphs $C_{4m} \Theta\, C_4$ for each $m \geq 1$ are odd harmonious.

Proof

Let $G = C_{4m} \Theta\, C_4$ be described as indicated in Figure (7)

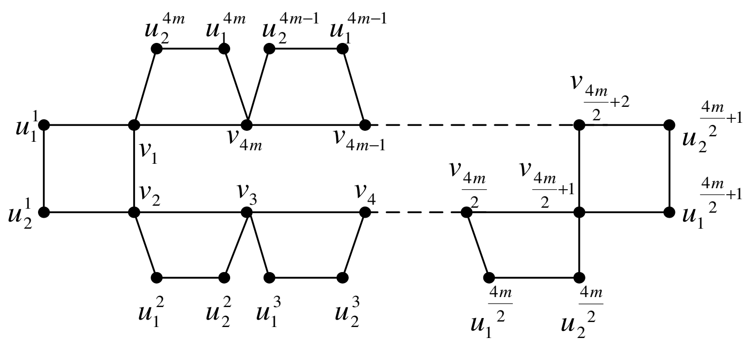


Figure (7)

It is clear that the number of edges of the graph $C_{4m} \Theta\, C_4$ is 16*m*. We define the labeling function

$f : V(G) \rightarrow \{0, 1, 2, \ldots, 32m - 1\}$

as follows:

$$f(v_i) = \begin{cases} 4(i-1), & (i \text{ odd}) \quad 1 \leq i \leq 4m-1 \\ 4i-3, & (i \text{ even}) \quad 2 \leq i \leq \dfrac{4m}{2}, \\ 4i+1, & (i \text{ even}) \quad \dfrac{4m}{2}+2 \leq i \leq 4m, \end{cases}$$

Now, for labeling the remaining vertices $u_i^1, u_i^2,\ 1 \leq i \leq 4m$ we consider the following cases:

Case(i): when $1 \le i \le \dfrac{4m}{2}$

$$f(u_i^1) = \begin{cases} 4i-3, & i = 1,3,5,\ldots,\dfrac{4m}{2}-1, \\ 4i-4, & i = 2,4,6,\ldots,\dfrac{4m}{2}, \end{cases}$$

$$f(u_i^2) = \begin{cases} 4i-2, & i = 1,3,5,\ldots,\dfrac{4m}{2}-1, \\ 4i-1, & i = 2,4,6,\ldots,\dfrac{4m}{2}, \end{cases}$$

Case(ii): when $\dfrac{4m}{2}+1 \le i \le 4m$

$$f(u_i^1) = \begin{cases} 4i+1, & i = \dfrac{4m}{2}+1,\dfrac{4m}{2}+3,\ldots,4m-1, \\ 4i-4, & i = \dfrac{4m}{2}+2,\dfrac{4m}{2}+4,\ldots,4m, \end{cases}$$

$$f(u_i^2) = \begin{cases} 4i-2, & i = \dfrac{4m}{2}+1,\dfrac{4m}{2}+3,\ldots,4m-1, \\ 4i+3, & i = \dfrac{4m}{2}+2,\dfrac{4m}{2}+4,\ldots,4m, \end{cases}$$

It follows that f is an odd harmonious labeling for $C_{4m}\Theta\, C_4$. Hence $C_{4m}\Theta\, C_4$ is an odd harmonious graph.

Example 2.11. An odd harmonious labeling of the $C_8\Theta\, C_4$, is shown in Figure (8).

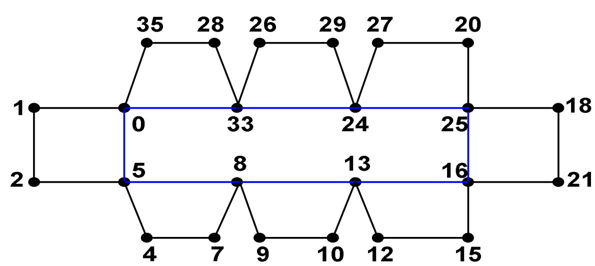


Figure (8): The graph $C_8\Theta\, C_4$ with its odd harmonious labeling.

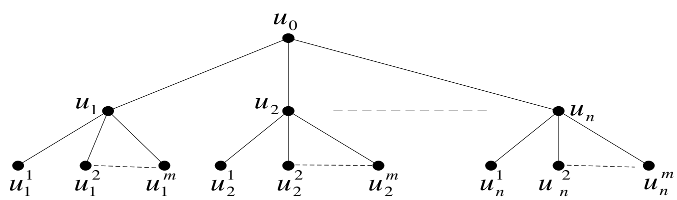
Let G_1 and G_2 be two disjoint graphs. The corona $(G_1\, \Theta\, G_2)$ of G_1 and G_2 is the graph obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 , and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

Theorem 2.12

The graphs $S_n \odot \overline{K_m}$, for $n, m \geq 1$ are odd harmonious graph.

Proof

Let $S_n \odot \overline{K_m}$, be described as indicated in Figure(9):



Figure(9)

It is clear that the number of edges of the graph $S_n \odot \overline{K_m}$ **,** is $q = n(m+1)$. We define the labeling function

f : V(G) $\rightarrow$ *{0, 1, 2, …, 2n(m+1) - 1}*
as follows:

$$f(u_0) = 0,$$
$$f(u_i) = 2i - 1, \quad 1 \leq i \leq n,$$
$$f(u_i^j) = 2q - (2m+2)(i-1) - 2j, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m,$$

Above defined labeling pattern exhausts all possibilities and the graph under consideration admits odd harmonious labeling.

Example 2.13. An odd harmonious labeling of the graph $S_3 \odot \overline{K_3}$, is shown in Figure (10).

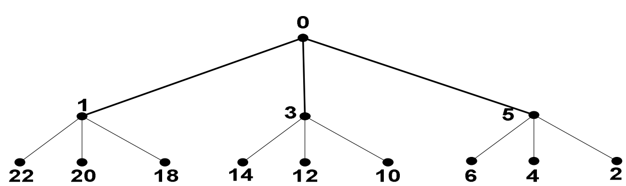


Figure (10): The graph $S_3 \odot \overline{K_3}$ with its odd harmonious labeling.

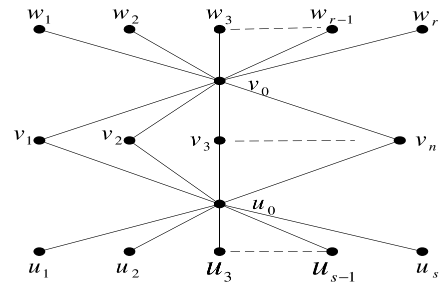
The graph $K_{2,n}$ (*r, s*) obtained from $K_{2,n}$,($n \geq 2$) by adding *r* and *s* (r, s ≥ 1) pendent edges out from the two vertices of degree *n*.

Theorem 2.14

The graphs $K_{2,n}$ *(r, s)* are odd harmonious for all *n, r, s* ≥ 1.

Proof

Let $K_{2,n}$ (*r, s*) be described as indicated in Figure(11)



Figure(11)

The number of edges of the graph $K_{2,n}$ (r, s) is $2n + r + s$. We define the labeling function :

$$f : V(K_{2,n}(r, s)) \to \{0, 1, 2, \ldots 2(2n + r + s) - 1\}$$

as follows

$$f(w_i) = 2i - 1 \quad , \quad 1 \le i \le r$$
$$f(v_0) = 0$$
$$f(v_i) = 2(i + r) - 1 \quad , \quad 1 \le i \le n$$
$$f(u_0) = 2n$$
$$f(u_i) = 2(r + n + i) - 1 \quad , \quad 1 \le i \le s$$

The edge labels will be as follows:

- The vertices v_0 and w_i , $1 \le i \le r$, induce the edge labels $\{1, 3, 5, \ldots, 2r - 1\}$.
- The vertices v_0 and v_i , $1 \le i \le n$, induce the edge labels $\{2r + 1, 2n + 2r + 3, \ldots, 2(r + n) - 1\}$.
- The vertices u_0 and v_i , $1 \le i \le n$, induce the edge labels $\{2(r + n) + 1, 2(r + n) + 3, \ldots, 2(2n+r) - 1\}$.
- The vertices u_0 and u_i, $1 \le i \le s$, induce the edge labels $\{2(2n+r) + 1, 2(2n+r) + 3, \ldots, 2(2n+r+s) - 1\}$.

So we obtain the edge labels $\{1, 3, 5, \ldots, 2(2n + r + s) - 1\}$. Hence the graph G is odd harmonious.

Example 2.15. An odd harmonious labeling of the $K_{2,5}$ (3, 4), is shown in Figure (12).

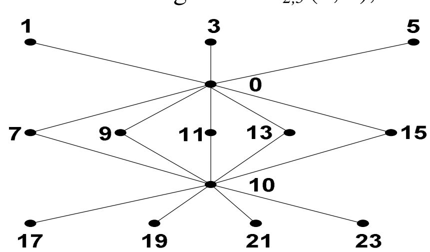


Figure (12): The graph $K_{2,5}$ (3, 4) with its odd harmonious labeling.

CONCLUSION

Since labelled graphs serve as practically useful models for wide-ranging applications such as communications network, circuit design, coding theory, radar, astronomy, X-ray and crystallogrophy, it is desired to have generalized results or results for a whole class, if possible. This work has presented several families of odd harmonious graphs. To investigate similar results for other graph families and in the context of different labeling techniques is open area of research.

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